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EQUIVALENCE OF GENERALIZED NETWORK  
AND GENERALIZED TRANSPORTATION PROBLEMS

Fred Glover, et al

Texas University

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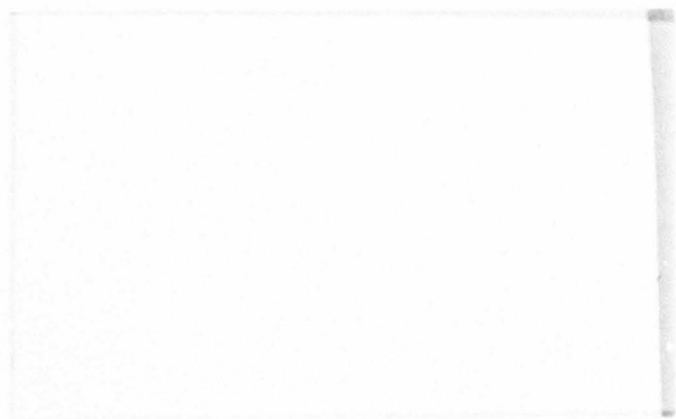
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TRANSPORTATION PROBLEMS

by

Fred Glover

D. Klingman

Al Napier

January 1973

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A. Charnes, Director  
Business-Economics Building, 512  
The University of Texas  
Austin, Texas 78712

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### Abstract

This paper shows that any generalized network problem can be transformed into a generalized transportation problem. Our approach extends earlier procedures for transforming pure network problems to ordinary transportation problems. Also we show that start and solution algorithms developed for certain classes of generalized network problems can be applied to any generalized network problem.

## 1. Introduction

The purpose of this paper is to show that any generalized network problem can be transformed into a generalized transportation problem. Our approach constitutes an extension of the procedures [ 6,7 ] for transforming ordinary ("pure") network problems into pure transportation problems. Since it has been shown [ 4 ] that any generalized network whose incidence matrix does not have full row rank is equivalent to a pure network problem, our results imply that such generalized networks are equivalent to ordinary transportation problems.

Our proposed transformation makes it possible to extend the range of application of those procedures [ 2,3 ] which have been developed for generalized network problems whose arc multipliers (arc amplification coefficients) are between 0 and 1 permitting them to be applied to any generalized network. (Note that it is not possible to develop a linear procedure for scaling an arbitrary generalized network problem to yield all arc multipliers in such a range.) This extension follows from the fact that any generalized transportation problem can be made to assume the desired form by simply scaling each column of the coefficient matrix to contain a 1 in its origin row and then dividing each destination row by the largest coefficient in the row. One computational aspect of these observations is that the procedure in [ 3 ] can be used to yield a dual feasible solution for any generalized network problem.

## 2. Transformation

For our purposes, a generalized network will be defined to consist of  $n$  nodes or junction points that are connected pairwise by a collection of  $m$  directed arcs (links). It is not necessary for all pairs of nodes to be joined.

For each arc  $(i,j)$  in the network, we also define the following items:

1.  $x_{ij}$  is the flow from node  $i$  to node  $j$ .
2.  $c_{ij}$  is the cost of sending a single unit of flow from node  $i$  to node  $j$ .
3.  $p_{ij}$  is the amplification (or attenuation) coefficient (or multiplier) on the flow from node  $i$  to node  $j$ ; i.e., if  $y$  units of flow leave node  $i$  then  $p_{ij}y$  units enter node  $j$ .

Letting  $N$  denote the set of arcs, a generalized network problem may be stated as:

Problem I

$$\text{Minimize } \sum_{(i,j) \in N} c_{ij} x_{ij} \quad (1.1)$$

subject to:

$$-\sum_{(i,j) \in N} x_{ij} + \sum_{(j,i) \in N} p_{ji} x_{ji} = d_i, \quad i = 1, 2, \dots, n \quad (1.2)$$

$$x_{ij} \geq 0, \quad (i,j) \in N, \quad p_{ij} > 0 \quad (1.3)$$

where  $d_i$  is the amount of supply (demand) at node  $i$ , where the supply (demand) at node  $i$  is denoted by a negative (positive)  $d_i$ .

Each node in a network can be classified as a source, sink, or transshipment node. A source node only has arcs emanating from it, while a sink node only has arcs entering it. A transshipment node has arcs both entering it and emanating from it. (Note: a transshipment node may have a supply or a demand.) In this formulation, we assume for simplicity that there are  $n_s$  sources,  $n_t$  transshipment nodes,  $n_d$  sinks and that these nodes are numbered in order from 1 to  $n$ . That is, the sources are numbered from 1 to  $n_s$ , the transshipment nodes from  $n_s + 1$  to  $n_s + n_t$  and the sinks from  $n_s + n_t + 1$  to  $n_s + n_t + n_d$ .



Our main result is that the following generalized transportation is equivalent to Problem I:

Problem II

$$\text{minimize } \sum_{(i,j+n_s) \in N} c'_{ij} x'_{ij} + \sum_{i=1}^{n_t} c'_{i+n_s,i} y_{i+n_s,i} \quad (2.1)$$

subject to:

$$\sum_{(i,j+n_s) \in N} x'_{ij} = a_i, \quad i = 1, \dots, n_s \quad (2.2)$$

$$\sum_{(i,j+n_s) \in N} x'_{ij} y_{i,i-n_s} = a_i, \quad i = n_s + 1, \dots, n_s + n_t \quad (2.3)$$

$$\sum_{(i,j+n_s) \in N} p'_{ij} x'_{ij} + y_{j+n_s,j} = b_j, \quad j = 1, \dots, n_t \quad (2.4)$$

$$\sum_{(i,j+n_s) \in N} p'_{ij} x'_{ij} = b_j, \quad j = n_t + 1, \dots, n_t + n_d \quad (2.5)$$

$$x'_{ij} \geq 0, \quad (i,j+n_s) \in N \quad (2.6)$$

$$y_{j+n_s,j} \geq 0, \quad j = 1, \dots, n_t \quad (2.7)$$

where  $c'_{ij} = c_{i,j+n_s}$ ,  $p'_{ij} = p_{i,j+n_s}$  for  $(i,j+n_s) \in N$

$$c'_{i+n_s,i} = 0 \text{ for } i = 1, 2, \dots, n_t$$

$$a_i = -d_i \text{ for } i = 1, \dots, n_s$$

$$a_i = -d_i + B \text{ for } i = n_s + 1, \dots, n_s + n_t$$

$$b_j = B \text{ for } j = 1, \dots, n_t$$

$$b_j = d_{j+n_s} \text{ for } j = n_t + 1, \dots, n_t + n_d$$

and B is a buffer to be specified later.

A succinct way of describing the transformation procedure for obtaining Problem II from Problem I is the following:

1. Designate an origin for each source  $i$  of the network and let the supply value  $a_i$  of this origin  $i$  be the negative of the amount of supply  $d_i$  at source  $i$ . ("Supplies" are positive in transportation formulations.)
2. Designate a destination  $j - n_s$  for each sink  $j$  of the network and let the demand value  $b_{j-n_s}$  of this destination be the amount of demand  $d_j$  at sink  $j$ .
3. For each transshipment node  $k$  designate an origin  $k$  and destination  $k - n_s$ . Let  $a_k = -d_k + B$  and  $b_{k-n_s} = B$ . ( $B$  is a buffer stock that must be large enough to insure that all  $y_{k,k-n_s}$  will be basic. A procedure for determining the appropriate value of  $B$  when all  $p_{ij} \geq 1$  or  $0 < p_{ij} \leq 1$  or  $p_{ij} > 0$  is examined in a later section.)
4. For each arc  $(i, j+n_s)$  of the generalized network introduce an arc  $(i, j)$  in the transportation problem with a cost  $c'_{ij}$  and multipliers  $p'_{ij}$  equal to the cost and multiplier associated with the original arc. In addition, for each transshipment node  $k$ , introduce an arc  $(k, k-n_s)$  in the transportation problem with a cost  $c'_{k,k-n_s}$  equal to zero and a multiplier  $p'_{k,k-n_s}$  equal to one. Let  $y_{k,k-n_s}$  denote the flow on this arc.

### 3. Equivalence

Theorem: Assume that  $0 < p_{ij} \leq 1$ ,  $(i, j) \in N$  and let  $B = - \sum_{\{i: d_i < 0\}} d_i$ . The solution

$x_{ij}$ ,  $(i, j) \in N$  is feasible (optimal) for Problem I if and only if the solution

$x'_{i, j-n_s} = x_{ij}$ ,  $(i, j) \in N$  and  $y_{i, i-n_s} = a_i - \sum_{(i, j+n_s) \in N} x'_{ij}$ ,  $i = n_s+1, \dots, n_s+n_t$  is

feasible (optimal) for Problem II. (Furthermore, the theorem is valid if the objective is to maximize the functionals (1.1) and (2.1).)

#### Proof:

It is apparent that the functionals for Problems I and II will have the

same value for the solution as indicated. Thus, to prove the theorem, it suffices simply to prove the feasibility assertion.

First assume that the solution  $x_{ij}, (i,j) \in N$  is feasible for Problem I.

We will show that  $x'_{i,j-n_s} = x_{ij}, (i,j) \in N$  and  $y_{i,i-n_s} = a_i - \sum_{(i,j+n_s) \in N} x'_{ij}, i=n_s+1, \dots, n_s+n_t$  is feasible for Problem II. By assumption  $x'_{i,j-n_s}, (i,j) \in N$  satisfies (2.6) and clearly by definition  $y_{i,i-n_s}$  and  $x'_{i,j-n_s}$  satisfy (2.3).

Note that for  $i = 1, \dots, n_s$ , (1.2) is equal to  $-\sum_{(i,j) \in N} x_{ij} = d_i$ . Since  $x'_{i,j-n_s} = x_{ij}$  and  $x_{ij}$  is a feasible solution for Problem I, then  $\sum_{(i,j) \in N} x'_{i,j-n_s} = -d_i$  or

$\sum_{(i,j+n_s) \in N} x'_{ij} = a_i, i=1, \dots, n_s$ ; thus (2.2) is satisfied. Similarly, for

$i = n_s+n_t+1, \dots, n$  (1.2) is equal to  $\sum_{(j,i) \in N} p_{ji} x_{ji} = d_i$ . From  $x'_{i,j-n_s} = x_{ij}$

and the feasibility of  $x_{ij}$  for Problem I we have  $\sum_{(j,i) \in N} p_{ji} x'_{j,i-n_s} = d_i$  or

$\sum_{(j,i) \in N} p'_{j,i-n_s} x'_{j,i-n_s} = d_i$  for  $i = n_s + n_t + 1, \dots, n$ . Setting  $k = i - n_s$  and rewriting we obtain  $\sum_{(j,k+n_s) \in N} p'_{jk} x'_{jk} = b_k (=d_{k+n_s})$ . Thus  $x'_{ij}$  satisfies (2.5).

The equality  $y_{i,i-n_s} = a_i - \sum_{(i,j+n_s) \in N} x'_{ij}, i=n_s+1, \dots, n_s+n_t$  can be

rewritten as  $y_{i,i-n_s} = -d_i + B - \sum_{(i,j) \in N} x_{ij}$ , and thus we have from (1.2)

$y_{i,i-n_s} + d_i - B + \sum_{(j,i) \in N} p_{ji} x_{ji} = d_i; i = n_s + 1, \dots, n_s + n_t$  or  $\sum_{(j,i) \in N} p_{ji} x_{ji} + y_{i,i-n_s} = B, i = n_s + 1, \dots, n_s + n_t$ . Setting  $k = i - n_s$  yields

$\sum_{(j,k+n_s) \in N} p_{j,k+n_s} x_{j,k+n_s} + y_{k+n_s,k} = B, k = 1, \dots, n_t$ . Since

$p'_{ij} = p_{i,j+n_s}, x'_{ij} = x_{i,j+n_s}$  and  $b_j = B$  for  $j=1, \dots, n_t$  the solution  $x'_{ij}$  satisfies (2.4).

Because all of the  $0 < p_{ij} \leq 1$ , the flow out of any transshipment node  $i$

cannot be greater than the total supply (B) less  $\max(d_i, 0)$ . Thus  $y_{i, i-n_s} = -d_i + B - \sum_{(i,j) \in N} x_{ij}$  is nonnegative for  $i = n_s + 1, \dots, n_s + n_t$ . Therefore (2.7)

is satisfied and this completes the first half of the proof.

Next, assume  $x'_{ij}$  is feasible for Problem II. We must now show that this solution is feasible for Problem I. It is immediately apparent that (1.3) is satisfied from (2.6); for  $i = 1, \dots, n_s$  (1.2) is satisfied from (2.2); and for  $i = n_s + n_t + 1, \dots, n$  (1.2) is satisfied from (2.5).

For  $k = n_s + 1, \dots, n_s + n_t$ , if equation  $k$  of (2.3) is subtracted from equation  $k - n_s$  of (2.4) we obtain

$$-\sum_{(k,j+n_s) \in N} x'_{kj} - y_{k, k-n_s} + \sum_{(i,k) \in N} p'_{i, k-n_s} x'_{i, k-n_s} + y_{k, k-n_s} = b_{k-n_s} - a_k.$$

Since  $b_{k-n_s} = B$  and  $a_k = -d_k + B$  we have

$$-\sum_{(k,j) \in N} x'_{kj} + \sum_{(i,k) \in N} p_{ik} x'_{ik} = d_k, \quad k = n_s + 1, \dots, n_s + n_t. \quad \text{Thus (1.2) is satisfied}$$

for all nodes  $1, 2, \dots, n$ , and the proof is complete.

Corollary: Assume that  $p_{ij} \geq 1$ ,  $(i,j) \in N$  and let  $B = \sum_{\{i: d_i > 0\}} d_i$ . The solution

$x_{ij}$  is feasible (optimal) for Problem I if and only if the solution  $x'_{i, j-n_s} = x_{ij}$ ,  $(i,j) \in N$  and  $y_{i, i-n_s} = a_i - \sum_{(i,j+n_s) \in N} x'_{ij}$ ,  $i = n_s + 1, \dots, n_s + n_t$  is feasible (optimal) for Problem II. (Furthermore, the corollary is valid if the objective is to maximize the functionals (1.1) and (2.1)).

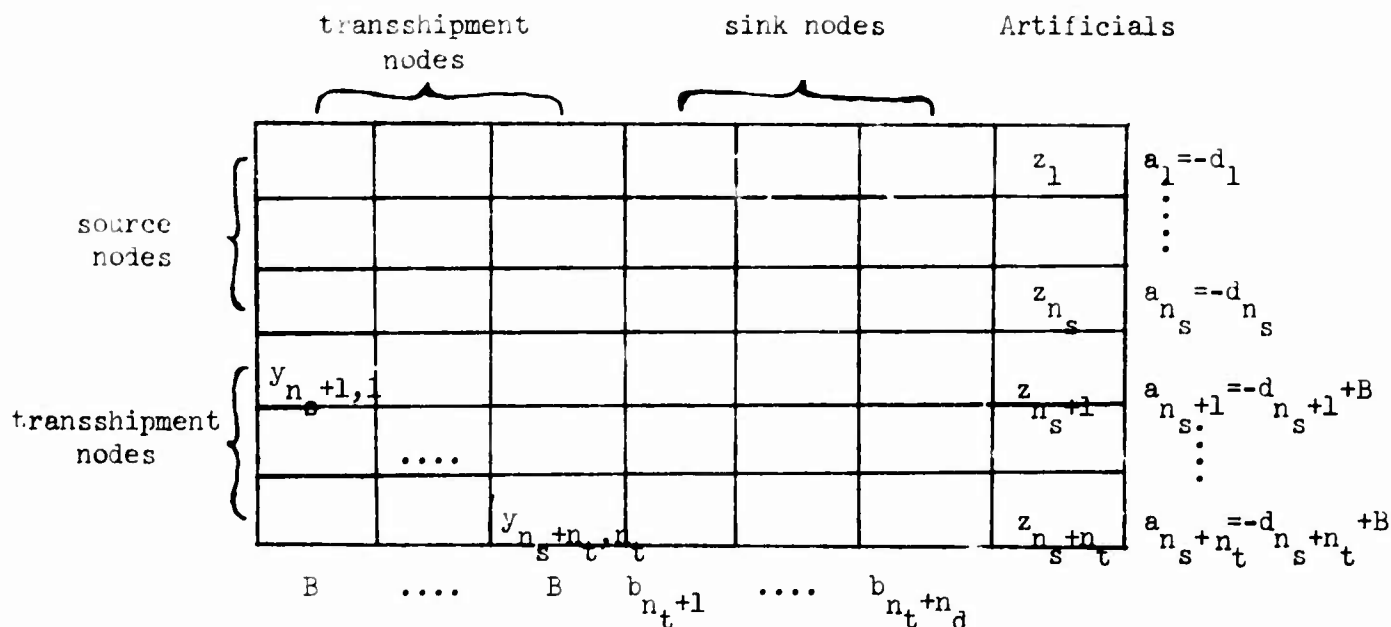
Proof: From the proof of the theorem, if  $x'_{ij}$  is a feasible solution for Problem II then  $x_{ij}$  is a feasible solution for Problem I regardless of the values of  $B$  and  $p_{ij}$ . Similarly, the proof of the theorem establishes that if  $x_{ij}$  is a feasible solution to Problem I, then  $x'_{ij}$  is a feasible solution

to Problem II if and only if  $y_{i,i-n_s} \geq 0$ . Thus it suffices simply to show that  $y_{i,i-n_s} \geq 0$ .

To do this, note that  $p_{ij} \geq 1$  for all  $(i,j)$  implies that the flow out of any transshipment node  $i$  cannot be greater than the total demand  $(B)$  less  $\max(0, d_i)$ . Otherwise, the amplification of the flow leaving the node would render the solution infeasible since this amplified flow could not be absorbed by the demand. Thus  $y_{i,i-n_s} = -d_i + B - \sum_{(i,j) \in N} x_{ij}$  is nonnegative for  $B$  equal to the total demand and this completes the proof.

If some of the multipliers  $p_{ij}$  are less than one and others are greater than one, then the problem may contain "creator" and/or "destructor" loops (See Jewell [ 5 ].) In such a case the solution region may be unbounded since arbitrarily large amounts of flow may be created and later destroyed. Thus, it is not possible to derive a sufficiently large value for the buffer  $B$  without assuming the nonexistence of creator or destructor loops. From a computational standpoint, however, it is not necessary to know a sufficiently large buffer size a-priori since the buffer can be successively increased until either an optimal solution is found to the generalized network or the problem is determined to be infeasible or unbounded. Moreover, this manipulation of the buffer can be done without interrupting the ordinary calculations and without shifting from a primal method to a dual method. This may be seen as follows.

Set the buffer at some positive value (i.e.,  $B > 0$ ) and try to solve the problem using a special purpose primal approach. First pick an artificial primal feasible starting basis containing the  $y_{i+n_s,i}$  variables. This can be done by considering the transportation tableau format for Problem II with a column of artificial variables  $z_j$  adjoined:



Set  $y_{n_s+i, i} = B, z_{n_s+i} = |d_{n_s+i}|, i=1, \dots, n_t$ , and subtract the artificial variable  $z_{n_s+i}$  from origin constraint  $n_s+i$  if  $d_{n_s+i} < 0$ ; otherwise add  $z_{n_s+i}$ . In addition if  $n_s \geq n_d$ , set  $x'_{i, n_t+i} = (1/p'_{i, n_t+i}) b_{n_t+i}, i=1, \dots, n_d$  and set  $z_i = |a_i - (1/p'_{i, n_t+i}) b_{n_t+i}|, i=1, \dots, n_d$  and subtract  $z_i$  if  $a_i - (1/p'_{i, n_t+i}) b_{n_t+i} < 0$ ; otherwise add  $z_i$ . Also set  $z_i = a_i, i=n_d+1, \dots, n_s$  and add it. (Note if  $x'_{i, n_t+i}, i=1, \dots, n_d$  does not exist then it is also an artificial variable and let  $p'_{i, n_t+i} = 1$ .)

If  $n_s \leq n_d$ , set  $x'_{i, n_t+i} = (1/p'_{i, n_t+i}) b_{n_t+i}, i=1, 2, \dots, n_s$  and  $x'_{n_s, n_t+i} = (1/p'_{n_s, n_t+i}) b_{n_t+i}, i=n_s+1, \dots, n_d$ . Further set  $z_i = |a_i - (1/p'_{i, n_t+i}) b_{n_t+i}|, i=1, 2, \dots, n_s-1$  and  $z_{n_s} = |a_{n_s} - \sum_{i=n_s}^{n_d} (1/p'_{i, n_t+i}) b_{n_t+i}|$ . Add  $z_i$  to the  $i^{th}$  origin if  $a_i - (1/p'_{i, n_t+i}) b_{n_t+i} \geq 0, i=1, 2, \dots, n_s-1$ ; otherwise subtract  $z_i$ . Similarly add or subtract  $z_{n_s}$ .

This is a basic artificial feasible solution since the set of the first  $n_s + 2n_t + n_d$  unit vectors is a subset of the span of the vectors associated with the basic variables. This can be easily seen by observing that the vector associated with the variable  $z_i$  consist of the first  $n_s + n_t$  unit vectors and subtracting these unit vectors from the other vectors associated with the other basic vectors yield the remaining unit vectors.

Using this artificial primal basic feasible solution consider performing a



Phase I optimization (i.e., minimizing the sum of the artificials). Throughout this minimization the buffer can be manipulated in a manner which enables the  $y_{i+n_s, i}$  variables to be kept basic. To see this first note that increasing the buffer  $B$  will increase only the basic  $y_{i+n_s, i}$  variables. Consequently, during Phase I whenever a variable  $x'_{ij}$  would enter the basis in place of some variable  $y_{n_s+i, i}$ , it is possible to increase the buffer sufficiently to prevent such a replacement from occurring. This is a consequence of the fact that the basis representation of any candidate to enter the basis must have a positive coefficient associated with at least one artificial variable. Thus, at the termination of Phase I all of the  $y_{n_s+i, i}$  variables will be basic. If any artificial variable is basic at a positive value then the generalized network problem is of course infeasible since increasing the buffer will only increase the  $y_{n_s+i, i}$  variables and thus not affect the artificial variables. (Specifically, there exist no buffer values for which the generalized transportation problem is feasible, consequently, the generalized network problem is infeasible.)

After completing Phase I and pivoting all zero-valued artificials out of the basis, it is either possible in Phase II to continue to keep the  $y_{n_s+i, i}$  variables basic by the same procedure of manipulating the buffer, or, the generalized network must be unbounded due to the fact that the incoming variable can be brought into the basis at an infinite amount by infinitely increasing the buffer. If the problem is not unbounded Phase II will terminate with a finite optimal solution to the generalized network since any increase in the buffer will not alter the solution value of the  $x'_{ij}$  to the corresponding transportation problem. (This illustrates that the constraints associated with the buffer act as "regularization constraints" as defined by Charnes [ 1 ].)

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